

## ACOUSTIC MODES IN SPHEROIDAL CAVITIES

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### Abstract

Oscillation modes of rapidly rotating stars have not yet been calculated with precision, rotational effects being generally approximated by perturbation methods. We are developing a numerical method able to account for the deformation of the star by the centrifugal force and, as a first step, we determined the acoustic modes of a uniform density spheroid and studied how the spheroid flatness affects these modes.

### 1 Introduction

The ratio of centrifugal and gravity forces being generally small in stars, the effect of rotation on gravito-acoustic stellar oscillations has been mostly studied with perturbation methods. Although fully justified in the context of helioseismology, this approach might not be accurate enough for rapidly rotating stars when the difference between the correct and approximated frequency becomes comparable with the accuracy of frequency measurements. We thus expect that new non-perturbative approaches will be necessary in the context of future asteroseismology missions (Mons, Corot, Eddington).

Such non-perturbative calculations should include the effects of the Coriolis and centrifugal forces on the wave motions as well as the effect of the centrifugal force on the equilibrium state of the star. This latter effect breaks down the spherical symmetry of the cavity within which acoustic modes resonate and this introduces mathematical difficulties since the related eigenvalue problem is no longer fully separable (except in the particular case of spheroids).

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In this paper, we present a numerical method able to solve this type of problem. Before it is applied to a realistic stellar model, we tested it in a simpler case. We choose the homogeneous spheroid because an alternative method exists in this case. Let also recall that stationary solutions of a self-gravitating and uniformly rotating gas of constant density are spheroids (the so-called Maclaurin spheroids). Thus acoustic modes of homogeneous spheroids can be viewed as the oscillation modes of Maclaurin spheroids in the high frequency limit (in order to neglect the effect of Coriolis force on the oscillatory motions) and within the Cowling approximation (to neglect the effect of gravity on the modes). In the next sections, the formalism is presented and the result of the test as well as a first analysis of the flatness effects on the frequency spectrum are given.

## 2 The formalism

Oscillatory motions of small amplitude in a perfect fluid of uniform density are governed by the Helmholtz equation,

$$\Delta\hat{\psi} + k^2\hat{\psi} = 0, \quad (2.1)$$

where  $\psi = \hat{\psi} \exp(i\omega t)$  is the velocity potential ( $\mathbf{u} = \nabla\psi$ ) and  $k = \omega/c_s$ , the ratio of the mode frequency,  $\omega$ , and the sound speed,  $c_s$ . The boundary condition at the surface of the cavity is,  $\mathbf{n} \cdot \nabla\hat{\psi} = 0$ , where  $\mathbf{n}$  is a vector normal to the surface.

Except for the special cases of spherical or spheroidal cavities this eigenvalue problem is not fully separable. Then, for arbitrary axially symmetric surfaces, a 2D eigenvalue problem must be solved numerically. We first note that, for numerical reasons, it is preferable to apply the boundary conditions on a coordinate surface. A coordinate system must thus be chosen so that the surface be a surface of coordinate ( $\zeta = \text{cste}$ ). That is, if  $r = S(\theta)$  describes the surface, suitable coordinates are  $(\zeta = f(\zeta, \theta), \theta, \phi)$  where for some value  $\zeta_0$  the function  $f$  verifies  $r = f(\zeta_0, \theta) = S(\theta)$ . In this expressions  $(r, \theta, \phi)$  are the usual spherical coordinates. Following Bonazzola et al. (1998),  $f$  is then specified so that the regularity conditions at the center have a simple form. For spheroids, we used,

$$r = f(\zeta, \theta) = R_e \left[ \zeta + (3\zeta^4 - 2\zeta^6) \left( \frac{1}{\sqrt{1 + g(\epsilon) \cos^2 \theta}} - 1 \right) \right] \quad (2.2)$$

where  $R_e$  and  $R_p$  are respectively the semi-major axis and the semi-minor axis,  $\epsilon = 1 - R_p/R_e$  is the flatness and  $g(\epsilon) = \epsilon(2 - \epsilon)/(1 - \epsilon)^2$ . The eigenvalue problem is written with these new coordinates. By expanding the solution  $\hat{\psi}(\zeta, \theta, \phi)$  on spherical harmonics and projecting the equations on each spherical harmonic  $Y_\ell^m(\theta, \phi)$ , one obtains for each value of the azimuthal number  $m$  two sets of ODE of the variable  $\zeta$  coupling the coefficients of spherical harmonic expansion corresponding respectively to odd and even degree numbers, that is  $(\hat{\psi}_m^{m+2k}(\zeta), 0 \leq k < +\infty)$  and  $(\hat{\psi}_m^{m+2k+1}(\zeta), 0 \leq k < +\infty)$ . Then following a method developed by Rieutord and Valdettaro (1997) after Orszag (1971), the ODE are discretized on the Gauss-Lobato grid associated with Chebyshev polynomials. A set of linear equations

dependent of the parameter  $k$  results and non-trivial solutions of this system are found using either the QZ algorithm or the Arnoldi-Chebyshev algorithm.

To test this general method, we considered a spheroidal cavity as a simple alternative method exists in this case. This is because the eigenvalue problem is separable using the so-called oblate spheroidal coordinates  $(\xi, \eta, \phi)$  defined as  $(x = a \cosh \xi \sin \eta \sin \phi, y = a \cosh \xi \sin \eta \cos \phi, z = a \sinh \xi \cos \eta)$ , where  $0 \leq \xi < +\infty$ ,  $0 \leq \eta \leq \pi$  et  $0 \leq \phi \leq 2\pi$ . The package Linear Solver Builder developed by Rieutord and Valdettaro was also used to solve the eigenvalue problem written in this coordinate system.

Finally, we tested our results in the range of small flatness using a perturbation approach. Following the method described in Gough et al. (1990), we calculated analytically the first order effect of the flatness on the mode frequencies. It reads,

$$\tilde{\omega}_{n\ell m} = \tilde{\omega}_{n\ell}^0 + \frac{2\epsilon\tilde{\omega}_{n\ell}^0}{3(2\ell-1)(2\ell+3)} \left( \ell(\ell+1) - 3m^2 + 3 \frac{\ell(\ell+1) - 3m^2}{(\tilde{\omega}_{n\ell}^0)^2 - \ell(\ell+1)} \right), \quad (2.3)$$

where  $\tilde{\omega}_{n\ell}^0 = \omega_{n\ell}^0 R/c_s$  is the dimensionless eigenfrequency corresponding to the sphere of the same volume,  $R = R_e (1 - \epsilon)^{1/3}$ . It is given as the  $n^{\text{th}}$  root (by growing order) of the following equation,  $(\ell+1)J_{\ell+1/2}(x) - xJ_{\ell-1/2}(x) = 0$ ,  $1 \leq \ell < +\infty$ , where  $J_\ell$  are Bessel functions.

### 3 Results

We found that the general method and the method specific to spheroidal cavities provide the same frequencies with a high level of accuracy for arbitrary value of the flatness between 0 and 0.5. Moreover, as the flatness goes to zero, the frequencies converge towards the values given by the asymptotic analysis (Eq. 2.3). We are thus encouraged in using the general method described in this paper for future pulsation models which shall take into account the deformation of the star equilibrium by the centrifugal force.

We also studied how the flatness of the spherical cavity affect the frequency of acoustic modes. The first point is to understand why, for increased flatness, the frequency of some modes increases while the frequency of others decreases. Simple arguments concerning the region of mode propagation can explain these behaviors. For large degree ( $\ell \gg 1$ ) quantitative results can even be deduced from such arguments: Large degree axi-symmetric modes ( $\ell \gg 1, m = 0$ ) propagate in meridional planes just below the cavity surface. Therefore, the characteristic length involved in their quantization condition shall be the perimeter of the circle,  $2\pi R$ , for the sphere, and the perimeter of the ellipse for the spheroid. By contrast sectorial modes ( $m = \ell$ ) are confined towards the equator so that the characteristic length is the perimeter of the equatorial circle that is  $2\pi R$  for the sphere and  $2\pi R_e$  for the spheroid. The increase or decrease of the modes frequency should then be determined by the ratio of these characteristic lengths. This interpretation is

consistent with the first order asymptotic formula (Eq. 2.3) since in the large  $\ell$  limit, the frequency ratio  $\tilde{\omega}_{n\ell m}/\tilde{\omega}_{n\ell}^0$  equals the ratio of the characteristic lengths.

As a consequence of the differential effect of flatness on frequencies, modes with originally distinct frequency tends to reach identical frequencies when flatness increases. However, this is not allowed for the coupled modes and avoided crossing occur. As all the modes having identical azimuthal number and degree numbers of the same parity are coupled (see above), avoided crossings occur at a strong rate in such a way that for non negligible flatness most of the modes have experienced many avoided crossing.

Another interest of the present calculation is to assess the limit of validity of the perturbative approach. Note first that, for small rotation rates, the flatness of MacLaurin spheroids is proportional to the ratio of centrifugal and gravity forces. Thus first order effects in terms of flatness correspond to second order effect in terms of the rotation rate. We found that the difference between the correct and the approximated frequency depends on the type of modes. But it can be relatively important. For example, at  $\epsilon = 0.1$ , the relative difference for axisymmetric modes of degree  $\ell = 1$  grows with the radial order and for  $1 \leq n \leq 6$  it is comprised between 0.01 and 0.03. If we extrapolate such relative differences to a rapidly rotating  $\delta$  Scuti oscillating around  $200 \mu$  Hz, we find a frequency difference of 1 to  $3 \mu$  Hz that is one order of magnitude larger than the accuracy planned for Corot mission. Although a firm conclusion should await stellar oscillation models, the present estimate points towards the necessity of using non-perturbative method for spatial seismology of rapidly rotating stars.

We also noticed the apparition of new types of modes in the spheroidal geometry. They are easily identified in the context of ray theory of axi-symmetric modes. Whereas for spherical cavity, all caustics are concentric spheres, two type of caustics, concentric spheroids and hyperboloids, are possible within a spheroidal cavity. In this last case, the ray does not propagate in the equatorial regions comprised between the surface boundary and the hyperbolic caustic. With our calculation we effectively found modes which show hyperbolic like regions of very low amplitude around the equator.

## 4 Future work

The next step is to calculate the oscillations of a rotating polytropic star including the effect of Coriolis and centrifugal forces on the motions. Among the new features arising when considering these more realistic stellar models is the chaos of rays dynamics within non-spherical and non-spheroidal cavities. Astrophysical consequences of this chaotic behavior shall be investigated.

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